Obstructions to deforming curves on a 3-fold, I: A generalization of Mumford's example and an application to Hom schemes

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Abstract

We give a sufficient condition for a first order infinitesimal deformation of a curve on a 3-fold to be obstructed. As application we construct generically non-reduced components of the Hilbert schemes of uniruled 3-folds and the Hom scheme from a general curve of genus five to a general cubic 3-fold.

1 Introduction

We study the (embedded) deformation of a (smooth projective) curve C on a smooth projective 3-fold V under the presence of a certain pair of a smooth surface S and a smooth curve E such that $C, E \subset S \subset V$. In other words we study the Hilbert scheme $\mathrm{Hilb}^{sc} V$ of smooth curves on V with the help of intermediate surfaces. Let $\tilde{C}$ be a first order infinitesimal deformation of $C \subset V$. As is well known, $\tilde{C}$ determines a global section α of the normal bundle $N_{C/V}$. It also determines a cohomology class $\mathrm{ob}(\alpha) \in H^1(N_{C/V})$ such that $\tilde{C}$ lifts to a deformation over $\mathrm{Spec}\, k[t]/(t^3)$ if and only if $\mathrm{ob}(\alpha)$ is zero (§2.1). This $\mathrm{ob}(\alpha)$ is called the *(primary) obstruction* of α (or $\tilde{C}$). It is generally difficult to compute $\mathrm{ob}(\alpha)$ for given α. In this paper, we give a sufficient condition for $\mathrm{ob}(\alpha) \neq 0$ in terms of $\pi_S(\alpha)$, the *exterior component* of α (Theorem 1.6). Here π_S is the natural projection $N_{C/V} \to N_{S/V}\big|_C$.

In each of the following examples, the tangent space $t_{W,C}$ of the subvariety W is everywhere of codimension one in $H^0(N_{C/V})$, the tangent space of $\mathrm{Hilb}^{sc} V$. If $[C] \in W$ is general, then every $\alpha \in H^0(N_{C/V})$ not in $t_{W,C}$ satisfies the condition, hence is obstructed. Therefore $\mathrm{Hilb}^{sc} V$ is everywhere non-reduced along W.

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Example 1.1. Let V be the projective space $\mathbb{P}^3$, $S \subset \mathbb{P}^3$ a smooth cubic surface, $E \subset S$ a (-1)-$\mathbb{P}^1$ and $C \subset S$ a smooth member of the linear system $|4h+2E| \simeq \mathbb{P}^{37}$ on S. C is of degree 14 and genus 24. Such C's are parametrized by $W = W^{56} \subset \operatorname{Hilb}^{sc} \mathbb{P}^3$, an open subset of a $\mathbb{P}^{37}$-bundle over $|3H| \simeq \mathbb{P}^{19}$. Here H is a plane in $\mathbb{P}^3$ and h is its restriction to S.

Example 1.2. Let V be a smooth cubic 3-fold $V_3 \subset \mathbb{P}^4$, S its general hyperplane section, $E \subset S$ a (-1)-$\mathbb{P}^1$ and $C \subset S$ a smooth member of $|2h+2E| \simeq \mathbb{P}^{12}$. C is of degree 8 and genus 5. Such C's are parametrized by $W = W^{16} \subset \operatorname{Hilb}^{sc} V$, an open subset of $\mathbb{P}^{12}$-bundle over the dual projective space $\mathbb{P}^{4,\vee}$. Here h is the restriction to S of a hyperplane H of $\mathbb{P}^4$. (See §3.1 for details.)

For many uniruled 3-folds V, we can find a curve $C \subset V$ similar to the above examples. More precisely we have the following:

Theorem 1.3. *Suppose that E is a (-1)-$\mathbb{P}^1$ on S, $N_{E/V}$ is generated by global sections and $p_g(S) = h^1(N_{S/V}) = 0$. Then the Hilbert scheme* $\operatorname{Hilb}^{sc} V$ *of smooth curves on V contains infinitely many generically non-reduced (irreducible) components W_n $(n \in \mathbb{Z}_{\geq 0})$ with the following property:*

(a) *every member of W_n is contained in a deformation of S in V, and*

(b) *every general member of W_n has a first order infinitesimal deformation whose primary obstruction is nonzero.*

See Example 3.7 for 3-folds V with such S and E.

Mumford [8] proved the non-reducedness of $\operatorname{Hilb}^{sc} \mathbb{P}^3$ (Example 1.1) by a global argument but later Curtin [2] gave another proof by infinitesimal analysis of deformations. Recently Nasu [9] has simplified and generalized Curtin's proof. This theorem follows the line of these works. Vakil [11] has also shown that various moduli spaces have non-reduced components by a different method (*cf.* Remark 3.9).

For a given projective scheme X, the set of morphisms $f : X \to V$ has a natural scheme structure as a subscheme of the Hilbert scheme of $X \times V$. This scheme is called the *Hom scheme* and denoted by $\operatorname{Hom}(X, V)$. When we fix a projective embedding $V \hookrightarrow \mathbb{P}^n$, all the morphisms of degree d are parametrized by an open and closed subscheme, which we denote by $\operatorname{Hom}_d(X, V)$. Our Example 1.2 gives rise to a counterexample to the following problem on the Hom scheme:

Problem 1.4 ($k = \mathbb{C}$)**.** *Is every component of* $\operatorname{Hom}(X, V')$ *generically smooth for a smooth curve X with general modulus and a general member V' of the Kuranishi family of V?*

Let $\mathrm{Hom}_8(X, V_3)$ be the Hom scheme of morphisms of degree 8 from a general curve X of genus 5 to a smooth cubic 3-fold $V_3 \subset \mathbb{P}^4$. Its expected dimension equals 4 (*cf.* §4).

Theorem 1.5 (char $k = 0$)**.** *Assume that V_3 is either (moduli-theoretically) general or of Fermat type*

$$V_3^{\mathrm{Fermat}} : x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

Then $\mathrm{Hom}_8(X, V_3)$ has an irreducible component of expected dimension which is generically non-reduced.

In order to prove Theorem 1.3, we take a rational section v of the normal bundle $N_{S/V}$. Suppose that v has a pole only along a smooth curve $E \neq C$ and of order one, that is, $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$. Then the divisor $(v)_0$ of zero does not contain E as a component. The restriction $v\big|_C$ belongs to $H^0(N_{S/V}\big|_C) \subset H^0(N_{S/V}(E)\big|_C)$ if and only if

$$(v)_0 \cap E \geq C \cap E \tag{1.1}$$

as a divisor on E.

Theorem 1.6. *Let $C, E \subset S \subset V$ and $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$ be as above and assume that $(E^2) < 0$. If the following conditions are satisfied, then every first order infinitesimal deformation $\tilde{C}$ of $C \subset V$, or α, whose exterior component coincides with $v\big|_C$ is obstructed.*

(a) *The equality holds in* (1.1).

(b) *Let ∂ denote the coboundary map of the exact sequence of*

$$0 \longrightarrow N_{E/S} \longrightarrow N_{E/V} \longrightarrow N_{S/V}\big|_E \longrightarrow 0 \tag{1.2}$$

tensored with $\mathcal{O}_S(E)$. Then the image $\partial(v\big|_E)$ of $v\big|_E \in H^0(N_{S/V}(E)\big|_E)$ is nonzero in $H^1(N_{E/S}(E)) \simeq H^1(\mathcal{O}_E(2E))$.

(c) *The restriction map $H^0(S, \Delta) \to H^0(E, \Delta\big|_E)$ is surjective, where $\Delta := C - 2E + K_V\big|_S$.*

If $E \subset S$ is a (-1)-$\mathbb{P}^1$, $N_{E/V}$ is generated by global sections and $v\big|_E$ is a general member of $H^0(N_{S/V}(E)\big|_E)$, then the condition (b) is satisfied (Lemma 3.6). If E is a (-1)-$\mathbb{P}^1$, we have $(\Delta.E) = 0$ and hence (c) is equivalent to that $|\Delta - E| + E \neq |\Delta|$.

We prove Theorem 1.6 in §2 and Theorem 1.3 in §3. In the final section, we prove Theorem 1.5.

We work over an algebraically closed field k in arbitrary characteristic except for in §4. We denote by $(A.B)$ the intersection number of two divisors A and B on a surface. For a subscheme S of V and a sheaf $\mathcal{F}$ on V, we denote the *restriction map* $H^i(V, \mathcal{F}) \to H^i(S, \mathcal{F}\big|_S)$ by $\big|_S$.

2 Obstruction to deforming curves

2.1 General theory

Let C be a smooth closed subvariety of a smooth variety V. We denote the normal bundle $\mathcal{H}om(\mathcal{I}_C, \mathcal{O}_C)$ of C in V by $N_{C/V}$, where $\mathcal{I}_C$ is the ideal sheaf of C in V. An *(embedded) first order infinitesimal deformation* of $C \subset V$ is a closed subscheme $\tilde{C} \subset V \times \operatorname{Spec} k[t]/(t^2)$ which is flat over $\operatorname{Spec} k[t]/(t^2)$ and whose central fiber is $C \subset V$. Let $\mathcal{I}_{\tilde{C}}$ be the ideal sheaf of $\tilde{C}$, which is also flat over $k[t]/(t^2)$. The multiplication endomorphism of $\mathcal{O}_V \otimes k[t]/(t^2)$ by t induces a homomorphism $\mathcal{I}_{\tilde{C}} \to \mathcal{O}_{\tilde{C}}$, which factors through $\alpha : \mathcal{I}_C \to t\mathcal{O}_C \simeq \mathcal{O}_C$. Moreover, $\tilde{C}$ is recovered from the homomorphism $\alpha : \mathcal{I}_C \to \mathcal{O}_C$. In the sequel we identify $\tilde{C}$ with $\alpha \in H^0(N_{C/V})$.

The standard exact sequence

$$0 \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_C \longrightarrow 0 \tag{2.1}$$

induces $\delta : H^0(N_{C/V}) = \operatorname{Hom}(\mathcal{I}_C, \mathcal{O}_C) \to \operatorname{Ext}^1(\mathcal{I}_C, \mathcal{I}_C)$ as a coboundary map. Then $\tilde{C}$ lifts to a deformation over $\operatorname{Spec} k[t]/(t^3)$ if and only if

$$\operatorname{ob}(\alpha) := \delta(\alpha) \cup \alpha = \alpha \cup \mathbf{k}_{C,V} \cup \alpha \in \operatorname{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$$

is zero, where $\mathbf{k}_{C,V} \in \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{I}_C)$ is the extension class of (2.1). $\operatorname{ob}(\alpha)$ is called the *obstruction* of α (or $\tilde{C}$). Since both C and V are smooth, $\operatorname{ob}(\alpha)$ is contained in the subspace $H^1(N_{C/V}) \subset \operatorname{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$ (*cf.* [7, Chap. I, Proposition 2.14]).

The tangent space of the Hilbert scheme $\operatorname{Hilb} V$ at $[C]$ is isomorphic to $H^0(N_{C/V})$. If $\operatorname{Hilb} V$ is nonsingular at $[C]$, then every first order infinitesimal deformation of $C \subset V$ lifts to a deformation over $\operatorname{Spec} k[t]/(t^n)$ for any $n \geq 3$. If $\operatorname{ob}(\alpha) \neq 0$ for some $\alpha \in H^0(N_{C/V})$, then $\operatorname{Hilb} V$ is singular at $[C]$.

Let L be a line bundle on V and $\delta : H^0(C, L\big|_C) \to H^1(V, L \otimes \mathcal{I}_C)$, $u \mapsto u \cup \mathbf{k}_{C,V}$, the coboundary map of the exact sequence $L\otimes$(2.1). We denote the composite of δ and the restriction map $\big|_C : H^1(V, L \otimes \mathcal{I}_C) \to H^1(C, L\big|_C \otimes N_{C/V}{}^\vee)$ by

$$d_{C,L} : H^0(C, L\big|_C) \longrightarrow H^1(C, L\big|_C \otimes N_{C/V}{}^\vee). \tag{2.2}$$

If V is projective, C is a divisor and $L = \mathcal{O}_V(C)$, then

$$d_{C,\mathcal{O}_V(C)} : H^0(C, N_{C/V}) \longrightarrow H^1(C, \mathcal{O}_C) \tag{2.3}$$

is the tangential map of the natural morphism $C' \mapsto \mathcal{O}_C(C')$ from the Hilbert scheme of divisors $C' \subset V$ to the Picard scheme $\operatorname{Pic} C$.

2.2 Exterior component

From now on we assume that C is contained in a smooth divisor $S \subset V$. There exists a natural exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/V} \xrightarrow{\pi_S} N_{S/V}\big|_C \longrightarrow 0 \tag{2.4}$$

of normal bundles. In this article we compute not $\mathrm{ob}(\alpha)$ itself but its image by

$$H^1(\pi_S) : H^1(N_{C/V}) \longrightarrow H^1(N_{S/V}\big|_C).$$

We call the image the *exterior component* of $\mathrm{ob}(\alpha)$ and denote by $\mathrm{ob}_S(\alpha)$. Here we give another expression of $\mathrm{ob}_S(\alpha)$. Let

$$d_{C,\mathcal{O}_V(S)} : H^0(N_{S/V}\big|_C) \simeq H^0(\mathcal{O}_C(S)) \longrightarrow H^1(C, N_{C/V}{}^{\vee} \otimes N_{S/V}\big|_C)$$

be the map (2.2) for the line bundle $L = \mathcal{O}_V(S)$. We abbreviate this as d_C.

Lemma 2.1.

$$\mathrm{ob}_S(\alpha) = d_C(\pi_S(\alpha)) \cup \alpha,$$

where $\cup$ *is the cup product map*

$$H^1(C, N_{C/V}{}^{\vee} \otimes N_{S/V}\big|_C) \times H^0(C, N_{C/V}) \xrightarrow{\cup} H^1(N_{S/V}\big|_C). \tag{2.5}$$

Proof. For each $i = 0$ and 1, $H^i(\pi_S)$ is equal to the restriction to $H^i(N_{C/V}) \subset \mathrm{Ext}^i(\mathcal{I}_C, \mathcal{O}_C)$ of the cup product map $\mathrm{Ext}^i(\mathcal{I}_C, \mathcal{O}_C) \xrightarrow{\cup\iota} \mathrm{Ext}^i(\mathcal{I}_S, \mathcal{O}_C)$, where $\iota : \mathcal{I}_S \hookrightarrow \mathcal{I}_C$ is the natural inclusion. Recall that the coboundary map δ in §2.1 is also a cup product map with the extension class $\mathbf{k}_{C,V}$ of (2.1). Therefore we have

$$\pi_S(\mathrm{ob}(\alpha)) = \iota \cup (\alpha \cup \mathbf{k}_{C,V} \cup \alpha) = \pi_S(\alpha) \cup \mathbf{k}_{C,V} \cup \alpha = d_C(\pi_S(\alpha)) \cup \alpha. \qquad \square$$

Let $d_S : H^0(C, N_{S/V}) \to H^1(S, \mathcal{O}_S)$ be the map (2.3) for $S \subset V$. d_S and d_C are closely related by the following commutative diagram:

$$\begin{array}{ccc} H^0(N_{S/V}) & \xrightarrow{d_S} & H^1(\mathcal{O}_S) \\ & & \downarrow{\scriptstyle |_C} \\ \downarrow{\scriptstyle |_C} & & H^1(\mathcal{O}_C) \\ & & \downarrow{\scriptstyle H^1(\iota)} \\ H^0(N_{S/V}\big|_C) & \xrightarrow{d_C} & H^1(N_{C/V}{}^{\vee} \otimes N_{S/V}), \end{array} \tag{2.6}$$

where $\iota : \mathcal{O}_C \to N_{C/V}{}^{\vee} \otimes N_{S/V}$ is the inclusion induced by π_S of (2.4). In some cases the exterior component of $\mathrm{ob}(\alpha)$ depends only on that of α.

Lemma 2.2. *If $\pi_S(\alpha) \in H^0(N_{S/V}\big|_C)$ lifts to a global section v of $N_{S/V}$, then we have*

$$\mathrm{ob}_S(\alpha) = d_S(v)\big|_C \cup \pi_S(\alpha).$$

Proof. By the diagram (2.6) we have $d_C(\pi_S(\alpha)) = H^1(\iota)(d_S(v)\big|_C)$. By the commutative diagram

$$\begin{array}{ccccc}
d_C(\pi_S(\alpha)) & & \alpha & & \\
\Cap & & \Cap & & \\
H^1(N_{C/V}{}^\vee \otimes N_{S/V}) & \times & H^0(N_{C/V}) & \xrightarrow{\cup} & H^1(N_{S/V}\big|_C) \\
\uparrow{\scriptstyle H^1(\iota)} & & \downarrow{\scriptstyle \pi_S} & & \| \\
H^1(\mathcal{O}_C) & \times & H^0(N_{S/V}\big|_C) & \xrightarrow{\cup} & H^1(N_{S/V}\big|_C), \\
\Cup & & & & \\
d_S(v)\big|_C & & & &
\end{array}$$

whose first cup product map is (2.5), we have

$$\mathrm{ob}_S(\alpha) = d_C(\pi_S(\alpha)) \cup \alpha = d_S(v)\big|_C \cup \pi_S(\alpha)$$

in $H^1(N_{S/V}\big|_C)$ by Lemma 2.1. $\square$

2.3 Infinitesimal deformation with a pole

We assume that V is a 3-fold, $S \subset V$ is a smooth surface and E is a smooth curve on S with $(E^2) < 0$ as in Theorem 1.6. We denote the complemental open varieties $S \setminus E$ and $V \setminus E$ by S° and V°, respectively, and the map (2.3) for $S^\circ \subset V^\circ$ by $d_{S^\circ} : H^0(N_{S^\circ/V^\circ}) \to H^1(\mathcal{O}_{S^\circ})$. In this subsection we study the singularity of $d_{S^\circ}(v) \in H^1(\mathcal{O}_{S^\circ})$ along the boundary E for $v \in H^0(S, N_{S/V}(E))$ (an infinitesimal deformation with a pole). The pole of $d_{S^\circ}(v)$ is of order at most 2 and its principal part coincides with $\partial(v\big|_E)$ (Proposition 2.4).

Let $\iota : S^\circ \hookrightarrow S$ be the open immersion. Then $\iota_*\mathcal{O}_{S^\circ}$ contains $\mathcal{O}_S(nE)$ as a subsheaf for any $n \geq 0$. There exists a natural inclusion $\mathcal{O}_S \subset \mathcal{O}_S(E) \subset \cdots \subset \mathcal{O}_S(nE) \subset \cdots$ and $\iota_*\mathcal{O}_{S^\circ}$ is the inductive limit $\lim_{n\to\infty} \mathcal{O}_S(nE)$.

Lemma 2.3. *Let L be a line bundle on S. If $\deg(L\big|_E) \leq 0$, then*

$$H^1(S, L) \longrightarrow H^1(S^\circ, L\big|_{S^\circ})$$

induced by the inclusion $L \hookrightarrow L \otimes \iota_*\mathcal{O}_{S^\circ}$ is injective.

Proof. There exists an open affine finite covering $\mathfrak{U} = \{U_i\}_{i=1,\ldots,n}$ of S. Let $\mathbf{c} = \{c_{ij}\}_{1\leq i<j\leq n}$ be a 1-cocycle with coefficient L with respect to $\mathfrak{U}$ and γ_m its cohomology class in $H^1(S, L(mE))$ for every $m \geq 0$. If $\mathbf{c}$ is a 1-coboundary of $L\big|_{S^\circ}$, then $\mathbf{c}$ becomes

that of $L(mE)$, that is, $\gamma_m = 0$, for a sufficiently large m. Since $\deg(L\big|_E) \le 0$, we have $H^0(E, L(mE)\big|_E) = 0$ for $m \ge 1$. Hence

$$H^1(S, L((m-1)E)) \longrightarrow H^1(S, L(mE))$$

is injective. Therefore, γ_{m-1} is also 0. By induction γ_0 is zero in $H^1(S, L)$. $\square$

By the lemma, the natural map $H^1(\mathcal{O}_S(2E)) \to H^1(\mathcal{O}_{S^\circ})$ is injective. We identify $H^0(N_{S/V}(E))$ and $H^1(\mathcal{O}_S(2E))$ with their images in $H^0(N_{S^\circ/V^\circ})$ and $H^1(\mathcal{O}_{S^\circ})$, respectively.

Proposition 2.4. (1) $d_{S^\circ}(H^0(S, N_{S/V}(E))) \subset H^1(S, \mathcal{O}_S(2E))$.

(2) *Let d_S be the restriction of d_{S° to $H^0(S, N_{S/V}(E))$ and let ∂ be the coboundary map in Theorem 1.6. Then the diagram*

$$\begin{array}{ccc} H^0(S, N_{S/V}(E)) & \xrightarrow{d_S} & H^1(S, \mathcal{O}_S(2E)) \\ \big\downarrow{\scriptstyle |_E} & & \big\downarrow{\scriptstyle |_E} \\ H^0(E, N_{S/V}(E)\big|_E) & \xrightarrow{\partial} & H^1(E, \mathcal{O}_E(2E)) \end{array}$$

is commutative.

Proof. Let $\mathfrak{U} := \{U_i\}_{i \in I}$ be an affine open covering of V and let $x_i = y_i = 0$ be the local equation of E over U_i such that y_i defines S in U_i. Through the proof, for a local section t of a sheaf $\mathcal{F}$ on V, $\bar{t}$ denotes the restriction $t\big|_S \in \mathcal{F}\big|_S$ for conventions. Let D_{x_i} and $D_{\bar{x}_i}$ denote the affine open subsets of U_i and $U_i \cap S$ defined by $x_i \neq 0$ and $\bar{x}_i \neq 0$, respectively. Then $\{D_{\bar{x}_i}\}_{i \in I}$ is an affine open covering of S° since $D_{\bar{x}_i} = D_{x_i} \cap S = U_i \cap S^\circ$.

Let v be a global section of $N_{S/V}(E) \simeq \mathcal{O}_S(S)(E)$. Then the product $\bar{x}_i v$ is contained in $H^0(U_i, \mathcal{O}_S(S))$ and lifts to a section $s_i' \in \Gamma(U_i, \mathcal{O}_V(S))$ since U_i is affine. In particular, v lifts to the section $s_i := s_i'/x_i$ of $\mathcal{O}_{V^\circ}(S^\circ)$ over D_{x_i}. Then we have

$$\delta(v)_{ij} = s_j - s_i \qquad \text{in} \qquad \Gamma(D_{x_i} \cap D_{x_j}, \mathcal{O}_{V^\circ}(S^\circ))$$

for every i, j, where $\delta : H^0(\mathcal{O}_{S^\circ}(S^\circ)) \to H^1(\mathcal{O}_{V^\circ})$ is the coboundary map of

$$[0 \longrightarrow \mathcal{I}_{S^\circ} \longrightarrow \mathcal{O}_{V^\circ} \longrightarrow \mathcal{O}_{S^\circ} \longrightarrow 0] \otimes \mathcal{O}_{V^\circ}(S^\circ)$$

in §2.1. Since v is a global section of $\mathcal{O}_{S^\circ}(S^\circ)$, $\delta(v)_{ij}$ is contained in $\Gamma(D_{x_i} \cap D_{x_j}, \mathcal{O}_{V^\circ})$. Moreover since $x_i s_i = s_i' \in \Gamma(U_i, \mathcal{O}_V(S))$ for every i, $f_{ij} := x_i x_j \delta(v)_{ij}$ is contained in $\Gamma(U_i \cap U_j, \mathcal{O}_V)$. Hence we have

$$d_{S^\circ}(v)_{ij} = (\delta(v)_{ij})\big|_{S^\circ} = \frac{\bar{f}_{ij}}{\bar{x}_i \bar{x}_j} \qquad \text{in} \qquad \Gamma(D_{\bar{x}_i} \cap D_{\bar{x}_j}, \mathcal{O}_{S^\circ}),$$

where $\bar{f}_{ij}$ is the restriction of $f_{ij} \in \mathcal{O}_{U_i \cap U_j}$ to $S \cap U_i \cap U_j$. By definition $\bar{f}_{ij}$ belongs to $\Gamma(S \cap U_i \cap U_j, \mathcal{O}_S)$. Hence $d_{S^\circ}(v)_{ij}$ is contained in $\Gamma(S \cap U_i \cap U_j, \mathcal{O}_S(2E))$. Thus we have proved (1).

Now we compute the image of $d_S(v) = d_{S^\circ}(v)$ by the restriction map $H^1(S, \mathcal{O}_S(2E)) \to H^1(E, \mathcal{O}_E(2E))$ regarding $\mathcal{O}_E(2E)$ as the quotient sheaf $\mathcal{O}_S(2E)/\mathcal{O}_S(E)$. For the computation, we need to consider the relation between the local equations $x_i = y_i = 0$ of E over U_i's. Since the two ideals $\mathcal{O}_{U_i} x_i + \mathcal{O}_{U_i} y_i$ and $\mathcal{O}_{U_j} x_j + \mathcal{O}_{U_j} y_j$ define the same ideal over $U_i \cap U_j$, there exist elements b_{ij} and c_{ij} of $\mathcal{O}_{U_i \cap U_j}$ satisfying $x_i = b_{ij} y_j + c_{ij} x_j$. Then we have

$$f_{ij} = x_i s'_j - x_j s'_i = (x_i - c_{ij} x_j) s'_j + (c_{ij} s'_j - s'_i) x_j = (b_{ij} y_j) s'_j + (c_{ij} s'_j - s'_i) x_j$$

and

$$\bar{f}_{ij} = \overline{(b_{ij} y_j) s'_j} + \overline{(c_{ij} s'_j - s'_i) x_j} \qquad \text{in} \qquad \Gamma(S \cap U_i \cap U_j, \mathcal{O}_S)$$

since $(b_{ij} y_j) s'_j$ belongs to $\mathcal{O}_{U_i \cap U_j}$. Hence we have

$$d_S(v)_{ij} = \frac{\bar{f}_{ij}}{\bar{x}_i \bar{x}_j} = \frac{\overline{b_{ij} y_j}}{\bar{x}_i} \cdot \frac{\overline{s'_j}}{\bar{x}_j} + \frac{\overline{c_{ij} s'_j - s'_i}}{\bar{x}_i} = \frac{\overline{b_{ij} y_j}}{\bar{x}_i} \cdot v + \frac{\overline{c_{ij} s'_j - s'_i}}{\bar{x}_i}$$

in $\Gamma(S \cap U_i \cap U_j, \mathcal{O}_S(2E))$, where $\overline{b_{ij} y_j}$ is a section of $\mathcal{O}_S(-S) \simeq {N_{S/V}}^\vee$ over $S \cap U_i \cap U_j$. Since

$$\overline{c_{ij} s'_j} - \overline{s'_i} = \bar{c}_{ij} \bar{x}_j v - \bar{x}_i v = 0 \qquad \text{in} \qquad \Gamma(S \cap U_i \cap U_j, \mathcal{O}_S(S)),$$

$c_{ij} s'_j - s'_i \in \mathcal{O}_{U_i \cap U_j}(S)$ is contained in $\mathcal{O}_{U_i \cap U_j}$. Hence $\overline{c_{ij} s'_j - s'_i}/\bar{x}_i$ is contained in $\Gamma(S \cap U_i \cap U_j, \mathcal{O}_S(E))$. On the other hand, the restriction of the 1-cochain $\left\{\overline{b_{ij} y_j}/\bar{x}_i\right\}_{i,j \in I}$ to E is a cocycle and represents the extension class $\mathbf{e} \in H^1(\mathcal{O}_E(-S+E))$ of the exact sequence (1.2). Therefore $d_S(v)\big|_E$ is equal to $\mathbf{e} \cup (v\big|_E) = \partial(v\big|_E)$, which shows (2). $\square$

2.4 Computation of obstructions

The purpose of this subsection is the proof of Theorem 1.6. Let v be a global section of $H^0(N_{S/V}(E))$ which satisfies the inequality (1.1). Let $\mathbf{k}_E = \mathbf{k}_{E,S}$ and $\mathbf{k}_C = \mathbf{k}_{C,S}$ be the extension classes of the exact sequences

$$0 \longrightarrow \mathcal{O}_S(-E) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_E \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0$$

on S, respectively. We regard $v\big|_C$ (resp. $v\big|_E$) as a global section of $N_{S/V}\big|_C$ (resp. $N_{S/V}(E-C)\big|_E$). Then we have the following:

Lemma 2.5.

$$v\big|_C \cup \mathbf{k}_C = v\big|_E \cup \mathbf{k}_E \qquad \text{in} \qquad H^1(N_{S/V}(-C)).$$

Proof. We have the following commutative diagram of $\mathcal{O}_S$-modules

$$\left[\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_S(-C-E) & \to & \mathcal{O}_S(-C)\oplus\mathcal{O}_S(-E) & \to & \mathcal{I}_{C\cap E} & \to & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle |_C} & & \\ 0 & \to & \mathcal{O}_S(-C-E) & \to & \mathcal{O}_S(-E) & \to & \mathcal{O}_C(-E) & \to & 0 \end{array}\right] \otimes N_{S/V}(E)$$

whose first row is the Koszul complex of $C\cap E$. By (1.1) the global section v belongs to $H^0(\mathcal{I}_{C\cap E}\otimes N_{S/V}(E))$. By the commutativity, the coboundary map

$$H^0(\mathcal{I}_{C\cap E}\otimes N_{S/V}(E)) \longrightarrow H^1(N_{S/V}(-C))$$

of the first row is equal to $(\cup \mathbf{k}_C)\circ\big|_C$ and similarly to $(\cup\mathbf{k}_E)\circ\big|_E$. $\square$

We need to consider the relation between the two maps d_C and d_S allowing pole along E. The diagram (2.6) becomes the *partially commutative* diagram

$$\begin{array}{ccccc} v & \in & H^0(N_{S/V}(E)) & \xrightarrow{d_S} & H^1(\mathcal{O}_S(2E)) \\ & & & & \big\downarrow{\scriptstyle |_C} \\ & & \big\downarrow{\scriptstyle |_C} & & H^1(\mathcal{O}_C(2Z)) \\ & & & & \big\downarrow{\scriptstyle H^1(\iota)} \\ & & H^0(N_{S/V}(E)\big|_C) & & H^1(N_{C/V}{}^\vee\otimes N_{S/V}(2Z)) \\ & & \cup & & \uparrow \\ u & \in & H^0(N_{S/V}\big|_C) & \xrightarrow{d_C} & H^1(N_{C/V}{}^\vee\otimes N_{S/V}), \end{array} \tag{2.7}$$

where Z is the scheme-theoretic intersection $C\cap E$. In other words, the commutativity holds only for $u\in H^0(N_{S/V}\big|_C)$ which has a lift $v\in H^0(N_{S/V}(E))$. More precisely, for such a pair u and v, we have

$$\overline{d_C(u)} = H^1(\iota)(d_S(v)\big|_C). \tag{2.8}$$

Here $\overline{*}$ denotes the image of $*\in H^1(C,\mathcal{F})$ (resp. $*\in H^1(S,\mathcal{F})$) in $H^1(C,\mathcal{F}(2Z))$ (resp. $H^1(S,\mathcal{F}(2E))$), where $\mathcal{F}$ is a vector bundle on C (resp. S).

Proof of Theorem 1.6 Let $\alpha\in H^0(N_{C/V})$ be as in the theorem. It suffices to show that the exterior component $\mathrm{ob}_S(\alpha)$ is nonzero in $H^1(N_{S/V}\big|_C)$. In fact, we show that its image $\overline{\mathrm{ob}_S(\alpha)}$ in $H^1(N_{S/V}(2E)\big|_C)$ is nonzero. The following generalizes Lemma 2.2 under the circumstances:

Step 1

$$\overline{\mathrm{ob}_S(\alpha)} = d_S(v)\big|_C \cup \pi_S(\alpha).$$

Proof. By Lemma 2.1, $\overline{\mathrm{ob}_S(\alpha)}$ is equal to the cup product $\overline{d_C(\pi_S(\alpha))} \cup \alpha$. By (2.8) $\overline{d_C(\pi_S(\alpha))}$ is equal to $H^1(\iota)(d_S(v)\big|_C)$. The rest of the proof is same as that of Lemma 2.2. By the commutative diagram

$$\begin{array}{ccccc}
H^1({N_{C/V}}^\vee \otimes N_{S/V}(2E)) & \times & H^0(N_{C/V}) & \stackrel{\cup}{\longrightarrow} & H^1(N_{S/V}(2E)\big|_C) \\
\uparrow{\scriptstyle H^1(\iota)} & & \downarrow{\scriptstyle \pi_S} & & \| \\
H^1(\mathcal{O}_C(2Z)) & \times & H^0(N_{S/V}\big|_C) & \stackrel{\cup}{\longrightarrow} & H^1(N_{S/V}(2E)\big|_C),
\end{array}$$

we have the required equation. $\square$

We relate $\mathrm{ob}_S(\alpha)$ with a cohomology class on E by Lemma 2.5:

Step 2

$$\overline{\mathrm{ob}_S(\alpha)} \cup \mathbf{k}_C = (d_S(v)\big|_E \cup v\big|_E) \cup \mathbf{k}_E$$

in $H^2(N_{S/V}(2E - C))$.

Proof. Since

$$\begin{array}{ccccc}
H^1(\mathcal{O}_C(2Z)) & \times & H^0(N_{S/V}\big|_C) & \stackrel{\cup}{\longrightarrow} & H^1(N_{S/V}(2E)\big|_C) \\
\uparrow{\scriptstyle |_C} & & \|{\scriptstyle \mathrm{id}} & & \|{\scriptstyle \mathrm{id}} \\
H^1(\mathcal{O}_S(2E)) & \times & H^0(N_{S/V}\big|_C) & \stackrel{\cup}{\longrightarrow} & H^1(N_{S/V}(2E)\big|_C) \\
\|{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cup \mathbf{k}_C} & & \downarrow{\scriptstyle \cup \mathbf{k}_C} \\
H^1(\mathcal{O}_S(2E)) & \times & H^1(N_{S/V}(-C)) & \stackrel{\cup}{\longrightarrow} & H^2(N_{S/V}(2E - C))
\end{array}$$

is commutative, we have a commutative diagram

$$\begin{array}{ccccc}
 & & \pi_S(\alpha) & & \overline{\mathrm{ob}_S(\alpha)} \\
 & & \cap & & \cap \\
H^1(\mathcal{O}_C(2Z)) & \times & H^0(N_{S/V}\big|_C) & \stackrel{\cup}{\longrightarrow} & H^1(N_{S/V}(2E)\big|_C) \\
\uparrow{\scriptstyle |_C} & & \downarrow{\scriptstyle \cup \mathbf{k}_C} & & \downarrow{\scriptstyle \cup \mathbf{k}_C} \\
H^1(\mathcal{O}_S(2E)) & \times & H^1(N_{S/V}(-C)) & \stackrel{\cup}{\longrightarrow} & H^2(N_{S/V}(2E - C)). \\
\cup & & & & \\
d_S(v) & & & &
\end{array} \tag{2.9}$$

Hence we have

$$\overline{\mathrm{ob}_S(\alpha)} \cup \mathbf{k}_C = (d_S(v)\big|_C \cup \pi_S(\alpha)) \cup \mathbf{k}_C = d_S(v) \cup (\pi_S(\alpha) \cup \mathbf{k}_C)$$

in $H^2(N_{S/V}(2E-C))$ by Step 1. There exists a commutative diagram

$$\begin{array}{ccccc} & & v\big|_E & & \\ & & \cap & & \\ H^1(\mathcal{O}_E(2E)) & \times & H^0(N_{S/V}(E-C)\big|_E) & \xrightarrow{\cup} & H^1(N_{S/V}(3E-C)\big|_E) \\ \uparrow \big|_E & & \downarrow \cup \mathbf{k}_E & & \downarrow \cup \mathbf{k}_E \\ H^1(\mathcal{O}_S(2E)) & \times & H^1(N_{S/V}(-C)) & \xrightarrow{\cup} & H^2(N_{S/V}(2E-C)), \\ \cup & & & & \\ d_S(v) & & & & \end{array} \tag{2.10}$$

similar to (2.9). Therefore by Lemma 2.5, we have

$$d_S(v) \cup (\pi_S(\alpha) \cup \mathbf{k}_C) = d_S(v) \cup (v\big|_E \cup \mathbf{k}_E) = (d_S(v)\big|_E \cup v\big|_E) \cup \mathbf{k}_E.$$

Thus we obtain the equation required. □

Step 3 Since $d_S(v)\big|_E = \partial(v\big|_E)$ by Proposition 2.4 (2), we obtain $d_S(v)\big|_E \neq 0$ by the assumption (b). Since $N_{S/V}(E-C)\big|_E$ is trivial by the assumption (a), we have $d_S(v)\big|_E \cup v\big|_E \neq 0$ in $H^1(N_{S/V}(3E-C)\big|_E) \simeq H^1(\mathcal{O}_E(2E))$. Consider the coboundary map

$$\cup \mathbf{k}_E : H^1(N_{S/V}(3E-C)\big|_E) \longrightarrow H^2(N_{S/V}(2E-C)),$$

which appears in (2.10). By the Serre duality, it is the dual of the restriction map

$$H^0(S, C + K_V\big|_S - 2E) \xrightarrow{\big|_E} H^0(E, (C + K_V - 2E)\big|_E),$$

which is surjective by the assumption (c). Hence the coboundary map $\cup \mathbf{k}_E$ is injective. Therefore we obtain $d_S(v)\big|_E \cup v\big|_E \cup \mathbf{k}_E \neq 0$ and hence by Step 2 we conclude that $\overline{\mathrm{ob}_S(\alpha)} \neq 0$ in $H^1(N_{S/V}(2E)\big|_C)$.

Thus we have proved Theorem 1.6.

3 Application to Hilbert schemes

In this section, we apply the result of the previous section to prove Theorem 1.3. We generalize Mumford's example (Example 1.1) and show that for many uniruled 3-folds V, their Hilbert schemes $\mathrm{Hilb}^{sc} V$ contain similar generically non-reduced components.

3.1 Dichotomy

We explain the detail of Example 1.2, which is a prototype of the non-reduced components constructed in §3.4. It is simpler than Mumford's example in applying Theorem 1.6. Let

C, E, S, V, h and W be as in Example 1.2. Then by $(C.E) = (2h + 2E.E) = 0$, the intersection $C \cap E$ is empty, which is the main reason for the simplicity. By adjunction, we have $\mathcal{O}_S(-K_S) \simeq \mathcal{O}_S(h) \simeq N_{S/V}$. By adjunction again, $N_{S/V}\big|_C$ and $N_{C/S}$ are isomorphic to $\mathcal{O}_C(K_C)$ and $\mathcal{O}_C(2K_C)$, respectively. By the exact sequence (2.4), $h^0(N_{C/V})$ is equal to

$$h^0(N_{C/S}) + h^0(N_{S/V}\big|_C) = h^0(2K_C) + h^0(K_C) = 12 + 5 = 17.$$

Hence the tangent space $t_{W,C}$ of W at $[C]$ is of codimension one in the tangent space $H^0(N_{C/V})$ of $\operatorname{Hilb}^{sc} V$. We have only the following two possibilities (*i.e.* dichotomy):

[A] W is an irreducible component of $(\operatorname{Hilb}^{sc} V)_{\mathrm{red}}$. Moreover $\operatorname{Hilb}^{sc} V$ is generically non-reduced along W.

[B] There exists an irreducible component W' of $\operatorname{Hilb}^{sc} V$ which contains W as a proper closed subset. $\operatorname{Hilb}^{sc} V$ is generically smooth along W.

We prove that the case [B] does not occur.

Proposition 3.1. *The Hilbert scheme of smooth curves on a smooth cubic* 3*-fold* V *contains a generically non-reduced component* $\tilde{W}$ *of dimension* 16 *such that* $(\tilde{W})_{\mathrm{red}} = W$*.*

Proof. Consider the exact sequence

$$0 \longrightarrow N_{S/V} \longrightarrow N_{S/V}(E) \longrightarrow N_{S/V}(E)\big|_E \longrightarrow 0.$$

Since $H^1(N_{S/V}) \simeq H^1(\mathcal{O}_S(h)) = 0$, there exists a rational section v of $N_{S/V}$ having a pole along E. By $H^1(N_{C/S}) \simeq H^1(2K_C) = 0$ and (2.4), there exists a first order infinitesimal deformation $\tilde{C} \subset V \times \operatorname{Spec} k[t]/(t^2)$ of $C \subset V$ whose exterior component $\pi_S(\alpha)$ coincides with $v\big|_C$. Since S is general, so is E. Hence, by [5, Proposition 1.3], the normal bundle $N_{E/V}$ is trivial. Moreover it is easily checked that the other conditions of Theorem 1.6 are satisfied. For example, since $v\big|_E \neq 0$ and the coboundary map ∂ is injective, we have (a). Since $C \sim -K_V\big|_S + 2E$, we have $\Delta = 0$ by definition. Therefore we have (b). Thus we conclude that $\tilde{C}$ is obstructed by the theorem. This implies [A]. $\square$

Remark 3.2. By a similar method, we can show the non-reducedness of $\operatorname{Hilb}^{sc} \mathbb{P}^3$ along $W = W^{56}$ for Mumford's example in arbitrary characteristic.

3.2 S-maximal family of curves

Let V be a smooth projective 3-fold and let S be a smooth surface in V. We introduce the notion of S-maximal families, which is analogous to s-maximal irreducible subsets

defined in [6] for $\mathrm{Hilb}^{sc}\mathbb{P}^3$. We assume that the Hilbert scheme $\mathrm{Hilb}\,V$ of V is nonsingular at $[S]$. Let U_S be the irreducible component passing through $[S]$ and let

$$V \times U_S \supset \mathcal{S} \xrightarrow{p_2} U_S$$

be the universal family over U_S. Let C be a smooth curve on S and assume that $\mathrm{Hilb}^{sc} S$ is nonsingular of expected dimension ($=\chi(N_{C/S})$) at $[C]$ (*i.e.* $H^1(N_{C/S}) = 0$). Then the Hilbert scheme $\mathrm{Hilb}^{sc}\mathcal{S}$, which is same as the relative Hilbert scheme of $\mathcal{S}/U_S$ is nonsingular at $[C]$. The first projection $p_1 : \mathcal{S} \to V$ induces the morphism $\mathrm{Hilb}^{sc}\mathcal{S} \to \mathrm{Hilb}^{sc} V$. Let $\mathcal{W}_{S,C}$ be the irreducible component of $\mathrm{Hilb}^{sc}\mathcal{S}$ passing through $[C]$. We call the image of $\mathcal{W}_{S,C}$ in $\mathrm{Hilb}^{sc} V$ the *S-maximal family of curves* containing C and denote it by $W_{S,C}$. We illustrate $W_{S,C}$ by the diagram

$$\begin{array}{ccccccc}
\mathcal{W}_{S,C} & \subset & \mathrm{Hilb}^{sc}\mathcal{S} & & \mathcal{S} & \text{(universal family)} \\
\downarrow & & \downarrow & \searrow & \downarrow & \\
W_{S,C} & \subset & \mathrm{Hilb}\,V & \supset & U_S. &
\end{array}$$

There exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{C/S} & \longrightarrow & N_{C/V} & \xrightarrow{\pi_S} & N_{S/V}\big|_C & \longrightarrow & 0 \\
& & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N_{C/S} & \longrightarrow & N_{C/\mathcal{S}} & \longrightarrow & \underbrace{N_{S/\mathcal{S}}\big|_C}_{\cong H^0(N_{S/V})\otimes\mathcal{O}_C} & \longrightarrow & 0.
\end{array}$$

Here the two horizontal sequences of normal bundles are exact. By the diagram, we have the following lemma.

Lemma 3.3. *The cokernel (resp. kernel) of the tangential map*

$$\kappa_{[C]} : H^0(N_{C/\mathcal{S}}) \longrightarrow H^0(N_{C/V}) \tag{3.1}$$

of $\mathcal{W}_{S,C} \to \mathrm{Hilb}^{sc} V$ *at* $[C]$ *is isomorphic to that of the restriction map* $H^0(N_{S/V}) \to H^0(N_{S/V}\big|_C)$.

3.3 Construction of obstructed curves

From now on we assume that the geometric genus $p_g(S)$ is zero and $H^1(N_{S/V}) = 0$. Let E be a (-1)-$\mathbb{P}^1$ on S whose normal bundle $N_{E/V}$ is generated by global sections. We denote by $\varepsilon : S \to F$ the blow-down of E from S.

Proposition 3.4. *Let Δ_1 be a very ample divisor on F. Then for each sufficiently large integer n, $\Delta_n := n\Delta_1$ satisfies the following conditions:*

[i] *the linear system* $\Lambda_n := |\varepsilon^*\Delta_n - K_V\big|_S + 2E|$ *on* S *contains a smooth connected member* C*,*

[ii] *the restriction map* $\Lambda_n \cdots \to \Lambda_n\big|_E$ *is surjective,*

[iii] $H^i(S, \varepsilon^*\Delta_n + E) = 0$ *for* $i = 1, 2$*,*

[iv] $H^1(S, \varepsilon^*\Delta_n - E) = 0$ *and*

[v] $H^1(C, N_{C/S}) = 0$.

In the next subsection, we will show that $\mathrm{Hilb}^{sc} V$ is non-reduced in a neighborhood of the corresponding point $[C]$.

Proof. We have $\chi(N_{E/V}) = \deg(-K_V\big|_E) = \deg N_{E/V} + 2$. Since $N_{E/V}$ is generated by global sections, we have $\deg N_{E/V} \geq 0$ and hence $\deg(-K_V\big|_E) \geq 2$. We have $(D.E) \geq -1$ for $D = E$, $-E$, $-K_V\big|_S + E$ and $-K_V\big|_S + 2E$. By the lemma below, there exists an integer m_1 such that for each $n \geq m_1$, all of the cohomology groups $H^i(\varepsilon^*(n\Delta_1) + E)$ $(i = 1, 2)$, $H^1(\varepsilon^*(n\Delta_1) - E)$, $H^1(\varepsilon^*(n\Delta_1) - K_V\big|_S + E)$ and $H^1(\varepsilon^*(n\Delta_1) - K_V\big|_S + 2E)$ vanish.

Put $e := \deg(-K_V\big|_S + 2E)\big|_E$. Then $e \geq 0$. Suppose that $e = 0$. Then there exists an integer m_2 such that for each $n \geq m_2$, $n\Delta_1 + \varepsilon_*(-K_V\big|_S + 2E)$ is very ample on F. Hence by the Bertini theorem (*cf.* [4, Chap. II, Theorem 8.18]), the linear system $|n\Delta_1 + \varepsilon_*(-K_V\big|_S + 2E)|$ contains a smooth connected member. Suppose that $e > 0$. Then there exists an integer m_2 such that for each $n \geq m_2$, $\Lambda_n = \left|\varepsilon^*(n\Delta_1) - K_V\big|_S + 2E\right|$ is base point free and $\varepsilon^*(n\Delta_1) - K_V\big|_S + 2E$ is ample. Then by the Bertini theorem again and [4, Chap. III Corollary 7.9], Λ_n contains a smooth connected member.

Assume that $n \geq \max\{m_1, m_2\}$ and let C be as in [i]. Then we obtain [ii], [iii] and [iv] by the choice of m_1. Finally we prove [v]. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow N_{C/S} \longrightarrow 0,$$

which induces

$$H^1(S, \mathcal{O}_S(C)) \longrightarrow H^1(C, N_{C/S}) \longrightarrow H^2(S, \mathcal{O}_S).$$

It follows from the choice of m_1 that $H^1(C) \simeq H^1(\varepsilon^*(n\Delta_1) - K_V\big|_S + 2E) = 0$. Since $p_g(S) = 0$, we have $H^1(N_{C/S}) = 0$. $\square$

Lemma 3.5. *Let* D *be a divisor on* S *with* $(D.E) \geq -1$*. Then there exists an integer* m_0 *such that for each* $i > 0$ *and each* $n \geq m_0$*, we have* $H^i(S, \varepsilon^*(n\Delta_1) + D) = 0$*.*

Proof. By assumption, D is linearly equivalent to $\varepsilon^*D' - jE$ for some divisor D' on F, where $j = (D.E) \geq -1$. If $j \geq 0$, by the Serre vanishing theorem, there exists an integer m_0 such that $H^i(S, \varepsilon^*(n\Delta_1) + D) \simeq H^i(F, \mathfrak{m}_p^j(D' + n\Delta_1)) = 0$ for each $n \geq m_0$, where $\mathfrak{m}_p$

is the maximal ideal at $p = \varepsilon(E)$. If $j = (D.E) = -1$, then we have $H^i(\mathcal{O}_E(D\big|_E)) = 0$. Hence by the exact sequence

$$[0 \longrightarrow \mathcal{O}_S(D - E) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_E(D\big|_E) \longrightarrow 0] \otimes \mathcal{O}_S(\varepsilon^*(n\Delta_1)),$$

the assertion follows from the case $j = 0$. □

3.4 Proof of Theorem 1.3

Let $C \in \Lambda_n$ be as in [i] of Proposition 3.4 for a sufficiently large integer n. By assumption and [v], $\operatorname{Hilb} V$, $\operatorname{Hilb}^{sc} S$ and $\operatorname{Hilb}^{sc} \mathcal{S}$ are nonsingular (of expected dimension) at $[S]$, $[C]$ and $[(C, S)]$, respectively. We consider the S-maximal family $W_{S,C}$ of curves containing C (*cf.* §3.2). Let $\kappa_{[C]} : H^0(N_{C/\mathcal{S}}) \to H^0(N_{C/V})$ be the tangential map of the morphism $\operatorname{Hilb}^{sc} \mathcal{S} \to \operatorname{Hilb}^{sc} V$ (*cf.* (3.1)). Then we have the following:

Claim 1 $\dim \operatorname{coker} \kappa_{[C]} = 1$.

Proof. By Lemma 3.3, the cokernel is isomorphic to $H^1(N_{S/V}(-C))$ since $H^1(N_{S/V}) = 0$. Since $N_{S/V}(-C) \sim K_S - K_V\big|_S - C \sim K_S - \varepsilon^*\Delta_n - 2E$, we have $H^1(N_{S/V}(-C)) \simeq H^1(\varepsilon^*\Delta_n + 2E)^\vee$ by the Serre duality. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(\varepsilon^*\Delta_n + E) \longrightarrow \mathcal{O}_S(\varepsilon^*\Delta_n + 2E) \longrightarrow \mathcal{O}_E(2E) \longrightarrow 0.$$

By the condition [iii], we have $H^1(\varepsilon^*\Delta_n + 2E) \simeq H^1(\mathcal{O}_E(2E)) \simeq k$. □

If $\alpha \in \operatorname{im} \kappa_{[C]}$, then the corresponding first order infinitesimal deformation $\tilde{C}$ of $C \subset V$ is realized as a member of $W_{S,C}$. Hence by Claim 1, the same dichotomy between [A] and [B] in §3.1 holds for $W := W_{S,C}$.

Claim 2 If $\alpha \notin \operatorname{im} \kappa_{[C]}$, then $\tilde{C}$ is obstructed.

Proof. We show that the exterior component $\pi_S(\alpha) \in H^0(N_{S/V}\big|_C)$ of α lifts to a rational section v of $N_{S/V}$ having a pole of order one along E. Consider a commutative diagram

$$\begin{array}{ccccc}
H^0(N_{S/V}) & \xrightarrow{|_C} & H^0(N_{S/V}\big|_C) & \twoheadrightarrow & H^1(N_{S/V}(-C)) \simeq k \\
\bigcap & & \bigcap & & \downarrow \\
H^0(N_{S/V}(E)) & \longrightarrow & H^0(N_{S/V}(E)\big|_C) & \longrightarrow & H^1(N_{S/V}(E - C)).
\end{array}$$

By the Serre duality and the condition [iii], we have $H^1(N_{S/V}(E-C)) \simeq H^1(\varepsilon^*\Delta_n+E)^\vee = 0$. Hence there exists $v \in H^0(N_{S/V}(E))$ such that $v\big|_C = \pi_S(\alpha)$. By the choice of α, v is not contained in $H^0(N_{S/V})$.

Now we check that the two assumptions (a) and (b) of Theorem 1.6 are satisfied. First we consider (a). Since $v\big|_C = \pi_S(\alpha)$ is contained in $H^0(N_{S/V}\big|_C)$, we have $(v)_0 \cap E \geq C \cap E$ as divisor on $E \simeq \mathbb{P}^1$. Note that

$$m := (C.E) = (-K_V\big|_S + 2E.E) = \deg(-K_V\big|_E) - 2 = \deg N_{E/V} = \deg N_{S/V}(E)\big|_E.$$

By the degree reason, we have $(v)_0 \cap E = C \cap E$. Since C is a general member of Λ_n, by the condition [ii], C meets E at general m points on E. Hence $v\big|_E$ is a general global section of $N_{S/V}(E)\big|_E$. Therefore we have (a) by Lemma 3.6 below. Consider the exact sequence $0 \to \mathcal{O}_S(\varepsilon^*\Delta_n - E) \to \mathcal{O}_S(\varepsilon^*\Delta_n) \to \mathcal{O}_E \to 0$ for (b). It follows from [iv] that the restriction map $H^0(S, \varepsilon^*\Delta_n) \to H^0(E, \mathcal{O}_E)$ is surjective. By Theorem 1.6, $\tilde{C}$ is obstructed. □

Lemma 3.6. *Let ∂ be the coboundary map in Theorem 1.6. If $N_{E/V}$ is generated by global sections and t is a general global section of $N_{S/V}(E)\big|_E$, then the image $\partial(t)$ is nonzero in $H^1(\mathcal{O}_E(2E))$.*

Proof. By assumption, we have $H^1(N_{E/V}(E)) = 0$ and hence ∂ is surjective. Since $t \in H^0(N_{S/V}(E)\big|_E)$ is general, it is not contained in the kernel of the coboundary map and hence $\partial(t) \neq 0$. □

Therefore, by Claim 2, we have [A] and hence we have proved Theorem 1.3.

3.5 Examples

If V is separably uniruled and $E \simeq \mathbb{P}^1$ sweeps out V, then $N_{E/V}$ is generated by global sections.

Example 3.7 (char $k = 0$)**.** The following V, S and E satisfy the assumption of Theorem 1.3.

(1) V is a Fano 3-fold whose anti-canonical class $-K_V$ is a sum $H + H'$ of two ample divisors H and H'. Then $|H|$ contains a smooth member S. If $(V, H) \not\simeq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(1)), (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(2)), (Q^3, \mathcal{O}_Q(1))$, then S contains a (-1)-$\mathbb{P}^1$ E.

(2) V has a $\mathbb{P}^1$-bundle structure $\pi : V \to F$ over a smooth surface F with $p_g = 0$ in Zariski topology. Let $S_1 \subset V$ be a rational section of π and S a smooth member of $|S_1 + \pi^*A|$ for a sufficiently ample divisor A on F. Then $\pi|_S : S \to F$ is birational but not isomorphic. Hence S contains a (-1)-$\mathbb{P}^1$ E.

(3) V has a $\mathbb{P}^2$-bundle structure $\pi : V \to C$ over a smooth curve C. Let $\mathcal{O}(1)$ be a relative tautological line bundle and S a smooth member of $|\mathcal{O}(2)(\pi^*A)|$ for a sufficiently ample divisor A on C. Then $\pi|_S : S \to C$ is a conic bundle with a reducible fiber. An irreducible component E of a reducible fiber is a (-1)-$\mathbb{P}^1$.

Remark 3.8. Deformations of the curve $C \subset \mathbb{P}^4$ in Example 1.2 in a smooth quintic 3-fold $V_5 \subset \mathbb{P}^4$ was studied as Voisin's example in Clemens-Kley [1]. They showed that the Hilbert scheme $\mathrm{Hilb}^{sc}_{8,5} V_5$ has an embedded component at $[C]$.

Remark 3.9. Recently Vakil [11] has shown that various moduli spaces, including the Hilbert schemes of space curves (but not including $\mathrm{Hilb}^{sc}\,\mathbb{P}^3$), satisfy "Murphy's law", *i.e.*, every singularity type of finite type over $\mathbb{Z}$ appears on the moduli spaces.

4 Application to Hom schemes

Let V be a smooth projective variety and X a (smooth projective) curve. It is well known that the Zariski tangent space of $\mathrm{Hom}(X, V)$ at $[f]$ is isomorphic to $H^0(X, f^*T_V)$ and the following dimension estimate holds:

$$\chi(f^*T_V) \le \dim_{[f]} \mathrm{Hom}(X, V) \le \dim H^0(X, f^*T_V), \tag{4.1}$$

where T_V is the tangent bundle of V. The lower bound $\chi(f^*T_V) = \deg f^*(-K_V) + n(1-g)$ is called the *expected dimension*, where $n = \dim V$ and g is the genus of X.

In this section we prove Theorem 1.5. First we recall a simple unprojection.

Lemma 4.1. *A smooth cubic surface $S \subset \mathbb{P}^3$ is isomorphic to the image of a smooth quartic del Pezzo surface $F = (2) \cap (2) \subset \mathbb{P}^4$ by a projection from a point on F.*

Proof. As is well known S contains a line E. Choose a homogeneous coordinates $(x_1 : x_2 : x_3 : x_4)$ of $\mathbb{P}^3$ such that E is defined by $x_1 = x_2 = 0$. Then the equation of S is $x_1 q(x) + x_2 q'(x) = 0$ for quadratic forms $q(x)$ and $q'(x)$. S is the image of the quartic del Pezzo surface $F : q(x) + x_2 y = q'(x) - x_1 y = 0$ in the projective 4-space $\mathbb{P}^4$ with coordinate $(x : y)$ from the point $(0 : 0 : 0 : 0 : 1) \in F$. □

Let X be a general curve of genus 5. The canonical model of X, that is, the image of $X \overset{K_X}{\hookrightarrow} \mathbb{P}^4$, is a general smooth complete intersection $q_1 = q_2 = q_3 = 0$ of three quadrics. Let p be a general point of the ambient space $\mathbb{P}^4$ and F_p the intersection of two quadrics q and q' which belong to the net of quadrics $\langle q_1, q_2, q_3 \rangle_k$ and pass through p. We denote the blow-up of F_p at p by $\pi_p : S_p \to F_p$. Then we have a commutative diagram:

$$\begin{array}{ccccc} X & \subset & F_p & \subset & \mathbb{P}^4 \\ \| & & \uparrow \pi_p & & \downarrow \Pi_p \\ C & \subset & S_p & \subset & \mathbb{P}^3, \end{array} \tag{4.2}$$

where C is the inverse image of X in S_p and Π_p is the projection from $p \in F_p \setminus X$. Since X belongs to the linear system $|-2K_F|$ on F_p, C belongs to $|\pi_p^*(-2K_F)| = |-2K_S + 2E|$ on S_p, where E is the exceptional curve of $S_p \to F_p$. Since q, q' and $p \in F_p$ are general, S_p is a general cubic surface by Lemma 4.1.

Let $\tilde{W}$ be the generically non-reduced component of $\mathrm{Hilb}^{sc}\, V_3$ in Proposition 3.1. Let $\varphi : \tilde{W} \to \mathfrak{M}_5$ be the classification morphism of $\tilde{W}$ to the moduli space of curves of genus

5. The fiber $\varphi^{-1}([X])$ at the point $[X] \in \mathfrak{M}_5$ is isomorphic to an open subscheme of $\mathrm{Hom}(X, V_3)$. We show that its Zariski closure T in $\mathrm{Hom}(X, V_3)$ satisfies the requirement of Theorem 1.5.

In the Fermat case, every general cubic surface is isomorphic to a hyperplane section of V_3^{Fermat} by the following theorem, for which we need $\operatorname{char} k = 0$.

Sylvester's pentahedral theorem ([10, Chap. §84], [3]) *Every general cubic form $F(y_0, y_1, y_2, y_3)$ of 4 variables is a sum $\sum_{i=0}^{4} l_i(y_0, y_1, y_2, y_3)^3$ of the cubes of 5 linear forms.*

Hence S_p is isomorphic to a hyperplane section of V_3^{Fermat} and C is a general member of $\tilde{W}^{\mathrm{Fermat}}$. Therefore, the classification morphism $\varphi^{\mathrm{Fermat}} : \tilde{W}^{\mathrm{Fermat}} \to \mathfrak{M}_5$ is dominant by the diagram (4.2) and the fiber T^{Fermat} is of dimension 4. Since $\mathfrak{M}_5$ is generically smooth, T^{Fermat} is generically non-reduced.

Theorem 1.5 for a general V_3 follows from the Fermat case by the upper semi-continuity theorem on fiber dimensions. $\square$

Theorem 1.5 holds true for a smooth V_3 if the answer to following is affirmative:

Problem 4.2. *Is the classification map*

$$(\mathbb{P}^4)^\vee \dashrightarrow \mathfrak{M}_{\mathrm{cubic}}, \qquad [H] \mapsto [H \cap V_3]$$

dominant for a smooth cubic 3-fold $V_3 \subset \mathbb{P}^4$?

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